THE ENTIRE CYCLIC COHOMOLOGY OF NONCOMMUTATIVE 3-SPHERES

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Abstract. In this paper, we compute the entire cyclic cohomology of noncommutative 3-spheres. First of all, we verify the Mayer-Vietoris exact sequence of entire cyclic cohomology in the framework of Fréchet *-algebras. Applying it to their noncommutative Heegaard decomposition, we deduce that their entire cyclic cohomology is isomorphic to the d'Rham homology of the ordinary 3-sphere with the complex coefficients.

1. Introduction

Since Connes [5] constructed a generalization of periodic cyclic cohomology which is called entire cyclic cohomology, its explicit computation is executed only for few examples (cf. [3, 5, 12]). As a matter of fact, their entire cyclic cohomologies are nothing but their periodic ones. Recently, the first named author [16] computed that of smooth noncommutative 2-tori, which have the same property cited above.

In this paper, we firstly formulate the Mayer-Vietoris exact sequence for entire cyclic cohomology, then we apply it to compute for smooth noncommutative 3-spheres. The key idea is based on Meyer's excision [14, 15] concerning the short exact sequences of Fréchet *-algebras to obtain a noncommutative Mayer-Vietoris exact sequence for entire cyclic cohomology. To use his excision, we need to construct a bounded linear section for a short exact sequence of Fréchet *-algebras. To ensure it, we reformulate the notion of metric approximation property in the framework of Fréchet *-algebras to solve the lifting problem (see [4]). We then use Baum, Hajac, Matthes and Szymańskis' method [1] for a Heegaard decomposition of smooth noncommutative 3-spheres since they pointed out an insufficient part of Matsumoto's costruction [13] in the case of C^*-algebras.

Under this circumstance, we conclude that the entire cyclic cohomology of noncommutative 3-spheres is the same as their periodic one.

Throughout this paper, θ is an irrational number in the open unit interval $(0, 1)$ and we use the notation $\mathbb{Z}_{\geq 0}$ for the set of all nonnegative integers.

2. Preliminaries

We prepare some notations and basic properties used throughout the paper. Let $\mathfrak{A}$ be a Fréchet *-algebra or F^*-algebra and denote by $C^\infty([0,1],\mathfrak{A})$ the set of all $\mathfrak{A}$-valued smooth functions on the closed unit interval $[0,1]$ with respect to Fréchet topology. Given an element $f \in C^\infty([0,1],\mathfrak{A})$ and an integer $n \geq 1$, we write by $f^{(n)}$ its n-th derivative of f at $t\,(0 < t < 1)$ and denote by $f_+^{(n)}(0), f_-^{(n)}(1)$ the n-th derivatives at 0 or 1 as follows:

$$f_+^{(n)}(0) = \lim_{t\to 0+} f^{(n)}(t)$$
$$f_-^{(n)}(1) = \lim_{t\to 1-0} f^{(n)}(t).$$

For $n = 0$, we write $f_+^{(0)}(0) = f(0), f_-^{(0)}(1) = f(1)$.

Definition 2.1. *For a F^*-algebra $\mathfrak{A}$, we define the suspension $S^\infty\mathfrak{A}$ of $\mathfrak{A}$ by*

$$S^\infty\mathfrak{A} = \{f \in C^\infty([0,1],\mathfrak{A}) \,|\, f_+^{(n)}(0) = f_-^{(n)}(1) = 0 \quad (n \geq 0)\}.$$

and we also define the cone $C^\infty\mathfrak{A}$ of $\mathfrak{A}$ by

$$C^\infty\mathfrak{A} = \{f \in C^\infty([0,1],\mathfrak{A}) \,|\, f_-^{(n)}(1) = 0 \quad (n \geq 0)\}.$$

Then we have the following short exact sequence:

$$0 \longrightarrow \mathfrak{I} \xrightarrow{\ i\ } C^\infty\mathfrak{A} \xrightarrow{\ q\ } \mathfrak{A} \longrightarrow 0,$$

where q is defined by $q(f) = f(0)$,

$$\mathfrak{I} = \{f \in C^\infty\mathfrak{A} \,|\, f(0) = 0\}$$

and i is the canonical inclusion. The map $s : \mathfrak{A} \to C^\infty\mathfrak{A}$ defined by

$$s(a)(t) = (1-t)a \quad (a \in \mathfrak{A}, t \in [0,1])$$

is a bounded linear section of q with respect to Fréchet topology. We need to know the entire cyclic cohomologies of $C^\infty\mathfrak{A}$ and $\mathfrak{I}$. We say that given two F^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, the map

$$\Phi : \mathfrak{A} \to C^\infty([0,1],\mathfrak{B})$$

is called a smooth homotopy if it is a bounded homomorphism with respect to Fréchet topology and two bounded homomorphisms $f, g : \mathfrak{A} \to \mathfrak{B}$ are smoothly homotopic if there exists a smooth homotopy Φ from $\mathfrak{A}$ to $\mathfrak{B}$ with $\Phi_0 = f, \Phi_1 = g$. A Fréchet algebra $\mathfrak{A}$ is smoothly homotopic to another one $\mathfrak{B}$ if there are two homomorphisms $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathfrak{A}$ such that $g \circ f$ (resp. $f \circ g$) is smoothly homotopic to the identity on $\mathfrak{A}$ (resp. $\mathfrak{B}$). According to Meyer [14], we know the homotopy invariance of entire cyclic cohomology in the framework of F^*-algebras:

Proposition 2.2 ([14])**.** *If two bounded homomorphisms are smoothly homotopic, then they induce the same map on the entire cyclic cohmology.*

We also mention the following lemma:

Lemma 2.3. *Let $S^\infty\mathfrak{A}, C^\infty\mathfrak{A}$ and $\mathfrak{I}$ be cited above, we then have that*

$$HE^*(C^\infty\mathfrak{A}) = 0, \quad HE^*(\mathfrak{I}) \simeq HE^*(S^\infty\mathfrak{A}).$$

Proof. By Proposition 2.2, it suffices to show that $C^\infty\mathfrak{A}$ is smoothly homotopic to 0 to obtain the former isomorphism. The map

$$F : C^\infty\mathfrak{A} \to C^\infty([0,1], C^\infty\mathfrak{A})$$

defined by

$$F_s(f)(t) = f(s + (1-s)t) \quad (f \in C^\infty\mathfrak{A}, \quad s, t \in [0,1])$$

gives a smooth homotopy on $C^\infty\mathfrak{A}$. Since F_0 is the identity on $C^\infty\mathfrak{A}$ and for any $f \in C^\infty\mathfrak{A}$,

$$F_1(f)(t) = f(1) = 0.$$

We know that $C^\infty\mathfrak{A}$ is smoothly homotopic to 0. For the latter one, we introduce the map $t \mapsto f(e^{1-1/t}) \quad (f \in C^\infty\mathfrak{A},\ t \in [0,1])$, which belongs to $S^\infty\mathfrak{A}$. Indeed, we note that for any $n \geq 1$, $\dfrac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of some functions such as

$$f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1).$$

In fact, for $n = 1$, we have that

$$\frac{d}{dt} f(e^{1-1/t}) = f^{(1)}(e^{1-1/t})\frac{e^{1-1/t}}{t^2}.$$

Suppose that the function $\dfrac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of fuctions

$$f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1),$$

then we deduce that

$$\begin{aligned}&\frac{d}{dt}\left(f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^m}\right)\\ &= f^{(k+1)}(e^{1-1/t})\frac{e^{(l+1)(1-1/t)}}{t^{m+2}} + l f^{(k)}\frac{e^{l(1-1/t)}}{t^{m+2}} - m f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^{m+1}},\end{aligned}$$

so is $\dfrac{d^{n+1}}{dt^{n+1}} f(e^{1-1/t})$. Because of the following equalities:

$$\lim_{t\to 0+} f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^m} = f_+^{(n)}(0)\cdot 0 = 0$$

$$\lim_{t\to 1-0} f^{(k)}(e^{1-1/t})\frac{e^{l(1-1/t)}}{t^m} = f_-^{(n)}(1) = 0,$$

for any $f \in \mathfrak{I}$, $k, l, m \geq 1$, the function $f(e^{1-1/t})$ belongs to $S^\infty\mathfrak{A}$. Let

$$r : \mathfrak{I} \to S^\infty\mathfrak{A}$$

be the map defined by

$$r(f)(t) = f(e^{1-1/t}) \quad (f \in \mathfrak{I},\, t \in [0,1])$$

and i the natural inclusion from $S^\infty\mathfrak{A}$ into $\mathfrak{I}$. For the proof that $r \circ i$ is smoothly homotopic to the identity on $S^\infty\mathfrak{A}$, we use the bounded homomorphism

$$G : S^\infty\mathfrak{A} \to C^\infty([0,1], S^\infty\mathfrak{A})$$

defined by

$$G_s(f)(t) = f(se^{1-1/t} + (1-s)t) \quad (f \in S^\infty\mathfrak{A},\, s, t \in [0,1])$$

which gives a smooth homotopy connecting $r \circ i$ and the identity on $S^\infty\mathfrak{A}$. We firstly show that $G_s(f) \in S^\infty\mathfrak{A}$ for any fixed $f \in S^\infty\mathfrak{A}, s \in [0,1]$. Since

$$\frac{d}{dt}G_s(f)(t) = f^{(1)}(se^{1-1/t} + (1-s)t)\left(\frac{s}{t^2}e^{1-1/t} + 1 - s\right),$$

we know that

$$\lim_{t\to 0+} \frac{d}{dt}G_s(f)(t) = f_+^{(1)}(0)\cdot(1-s) = 0$$
$$\lim_{t\to 1-0} \frac{d}{dt}G_s(f)(t) = f_-^{(1)}(1) = 0.$$

For general $n \geq 2$, we also see that

$$\lim_{t\to 0+} \frac{d^n}{dt^n}G_s(f)(t) = \lim_{t\to 1-0} \frac{d^n}{dt^n}G_s(f)(t) = 0.$$

The case for $n = 1$ has already been shown. It suffices to show that for $n \geq 2$, the fuction $\frac{d^n}{dt^n}G_s(f)(t)$ is a linear combination of fuctions like

$$f^{(k)}(se^{1-1/t} + (1-s)t)\frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1).$$

We now calculate that

$$\begin{aligned}
&\frac{d}{dt}f^{(k)}(se^{1-1/t} + (1-s)t)\frac{e^{l(1-1/t)}}{t^m} \\
&= f^{(k+1)}(se^{1-1/t} + (1-s)t)\frac{e^{l(1-1/t)}}{t^m}\left(\frac{s}{t^2} + 1 - s\right) \\
&\quad + f^{(k)}(se^{1-1/t} + (1-s)t)\left(\frac{le^{l(1-1/t)}}{t^{m+2}} - \frac{me^{l(1-1/t)}}{t^{m+1}}\right),
\end{aligned}$$

which completes the induction process. Moreover we see that $\frac{d^n}{dt^n}G_s$ is uniformly bounded on $[0,1]$ for each $n \geq 1$. We note that the function

$$t \mapsto \frac{e^{l(1-1/t)}}{t^m}$$

is bounded on $[0,1]$ and that $f^{(k)}$ is also bounded since $f \in C^\infty([0,1], \mathfrak{A})$. Hence G is a smooth homotopy connecting $r \circ i$ and the identity on $S^\infty\mathfrak{A}$ since $G_1 = r \circ i$

and G_0 is the identity on $S^\infty\mathfrak{A}$. Similarly, $i \circ r$ and the identity on $\mathfrak{I}$ are smoothly homotopic via the smooth homotopy defined by the same way as G, which implies that

$$HE^*(\mathfrak{I}) \simeq HE^*(S^\infty\mathfrak{A})$$

as desired. □

3. Toeplitz F^*-Algebras

In this section, we construct smooth Toeplitz algebras based on 1-torus and to analyze them. They could be viewd as a quantization of 2-disc (cf. [1, 11]). Let $\{z^n\}_{n\in\mathbb{Z}}$ be the orthonomal basis of the Hilbert space $L^2(T)$ of all square integrable functions on the 1-torus T, where $z^n(t) = t^n$ $(t \in T, n \in \mathbb{Z})$, and $H^2 = H^2(T)$ the Hardy space on T which is a closed subspace of $L^2(T)$ spanned by $\{z^n\}_{n\geq 0}$. For $f \in C^\infty(T)$ of all infinitely differentiable functions on T, in which we mean that the derivation is defined by

$$\frac{d}{dt}f(t) = \lim_{r\to 0}\frac{f(e^{2\pi ir}t) - f(t)}{r},$$

we define the operator T_f for $f \in C^\infty(T)$ by

$$T_f\xi = Pf\xi \quad (\xi \in H^2),$$

where P is the projection onto H^2. We consider the *-algebra $\mathcal{P}$ generated by T_{z^j} $(j \in \mathbb{Z})$, namely,

$$\mathcal{P} = \bigcup_{N\in\mathbb{Z}_{\geq 0}}\left\{\sum_{i_j\in\mathbb{Z},\,|i_j|\leq N} c_{i_1,\dots,i_n}T_{z^{i_1}}\dots T_{z^{i_n}}\,\middle|\, c_{i_1,\dots,i_n}\in\mathbb{C}, n\in\mathbb{Z}_{\geq 0}\right\}.$$

Since $T_fT_g - T_{fg}$ is a compact operator for any $f, g \in C^\infty(T)$ and T_f is compact if and only if $f = 0$ (cf. [8]), it is easily seen by induction that for any $T \in \mathcal{P}$, there is a unique $f \in C^\infty(T)$ and a unique compact operator S with $T = T_f + S$. Actually, if

$$T = \sum_{i_j\in\mathbb{Z},\,|i_j|\leq N} c_{i_1,\dots,i_n}T_{z^{i_1}}\dots T_{z^{i_n}} \in \mathcal{P},$$

then $T = T_f + S$, where

$$f = \sum_{i_j\in\mathbb{Z},\,|i_j|\leq N} c_{i_1,\dots,i_n}z^{i_1}\dots z^{i_n}$$

and the compact operator S is a linear combination of the operators of the form

$$T_{z^{l_1}}\cdots T_{z^{l_k}}(T_{z^n}T_{z^m} - T_{z^{n+m}})T_{z^{l'_1}}\cdots T_{z^{l'_{k'}}} \quad (l_1, \dots l_k, \dots l'_1, \dots l'_{k'}, n, m \in \mathbb{Z}).$$

We show that there exists a function $K_S(t,s) \in C^\infty(T^2)$ which is a polynomial of t, s and satisfies

$$(S\xi)(t) = \int_T K_S(t,s)\xi(s)ds. \quad (\xi \in H^2).$$

This function K_S is called the kernel function of S. Given $n, m \in \mathbb{Z}$, it is easily verified that

$$(T_{z^n}T_{z^m} - T_{z^{n+m}})\xi(t) = \left(\sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) \langle \xi \mid z^k \rangle z^k(t)$$
$$= \int_T \left(\sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) t^k s^{-k} \xi(s) ds,$$

where

$$\langle f \mid g \rangle = \int_T f(s)\overline{g(s)} ds \quad (f, g \in L^2(T))$$

is the usual inner product on $L^2(T)$. Then the kernel function $K_{T_{z^n}T_{z^m}-T_{z^{n+m}}}$ of $T_{z^n}T_{z^m} - T_{z^{n+m}}$ is a finitie sum of the fuctions $t^k s^{-k}$ since there exists a finite subset $I_{n,m} \subset \mathbb{Z}$ such that

$$K_{T_{z^n}T_{z^m}-T_{z^{n+m}}}(t,s) = \left(\sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) t^k s^{-k} = \pm \sum_{k \in I_{n,m}} t^k s^{-k}$$

(when $I_{n,m}$ is empty, we regard the function $K_{T_{z^n}T_{z^m}-T_{z^{n+m}}}(t,s) = 0$). Moreover, given $l \in \mathbb{Z}$, we compute that

$$K_{(T_{z^n}T_{z^m}-T_{z^{n+m}})T_{z^l}}(t,s) = \int_T K_{T_{z^n}T_{z^m}-T_{z^{n+m}}}(t,r) K_{T_{z^l}}(r,s) dr$$
$$= \pm \int_T \sum_{k \in I_{n,m}} t^k r^{-k} \sum_{k' \geq -l} r^{k'} s^{-k'} dr$$
$$= \pm \sum_{k' \geq -l} \sum_{k \in I_{n,m}} t^k s^{-k'} \int_T r^{k'-k} dr$$
$$= \pm \sum_{k \in I_{n,m},\, k \geq -l} t^k s^{-k},$$

which implies that the kernel function $K_{(T_{z^n}T_{z^m}-T_{z^{n+m}})T_{z^l}}$ is a polynomial of t, s. By the similar computation, it follows that for $l, m, n \in \mathbb{Z}$, the kernel function $K_{T_{z^l}(T_{z^n}T_{z^m}-T_{z^{n+m}})}$ is also a polynomial. Then, by the inductive argument, we have that the kernel functions

$$K_{T_{z^{l_1}} \cdots T_{z^{l_k}} (T_{z^n}T_{z^m}-T_{z^{n+m}}) T_{z^{l'_1}} \cdots T_{z^{l'_{k'}}}}$$

are also polynomials, which in particular belong to $C^\infty(T^2)$.

Let $\mathbb{K}^\infty$ be the set of all compact operators S such that there exists a function $K_S \in C^\infty(T^2)$ with the property that

$$(S\xi)(t) = \int_T K_S(t,s)\xi(s) ds \quad (\xi \in H^2,\, t \in T).$$

By the above argument, it follows that for each operator $T \in \mathcal{P}$, there exist a function $f \in C^\infty(T)$ and an operator $S \in \mathbb{K}^\infty$ with $T = T_f + S$. Since T_g is

compact if and only if $g = 0$, the function f and the operator S are uniquely determined. We define the seminorms $\{\|\cdot\|_{k,l,m}\}$ on $\mathcal{P}$ by

$$\|T_f + S\|_{k,l,m} = \|f^{(k)}\|_\infty + \|K_S^{(l,m)}\|_\infty \quad (k, l, m \in \mathbb{Z}_{\geq 0}),$$

where $f^{(k)}$ is the k-th derivative of f,

$$K^{(l,m)} = \frac{\partial^{l+m}}{\partial t^l \partial s^m} K(t, s) \quad (K \in C^\infty(T^2)),$$

and $\|\cdot\|_\infty$ mean the supremum norms on the corresponding function spaces.

Definition 3.1. *The smooth Toeplitz algebra $\mathcal{T}^\infty$ is defined by the completion of $\mathcal{P}$ with respect to the topology induced by the seminorms $\{\|\cdot\|_{k,l,m}\}$.*

Similarly as in the case of $\mathcal{P}$, we have that for any $T \in \mathcal{T}^\infty$, there exist a function $f \in C^\infty(T)$ and an operator $S \in \mathbb{K}^\infty$ with $T = T_f + S$. In fact, if $\{T_n\}_{n\geq 1} \subset \mathcal{P}$ converges to T with respect to the seminorms $\{\|\cdot\|_{k,l,m}\}$ with $T_n = T_{f_n} + S_n$, we compute that

$$\begin{aligned} \|f_n^{(k)} - f_{n'}^{(k)}\|_\infty &= \|T_{f_n} - T_{f_{n'}}\|_{k,0,0} \\ &\leq \|T_n - T_{n'}\|_{k,0,0} \to 0 \quad (n, n' \to \infty), \end{aligned}$$

for any $k \in \mathbb{Z}_{\geq 0}$, which ensures that there exists the function $f \in C^\infty(T)$ such that $f_n \to f$ with respect to the seminorms. Alternatively, since $\{S_n\}$ is also Cauchy, we have that for any $k, l, m \in \mathbb{Z}$,

$$\|S_n - S_{n'}\|_{k,l,m} = \|K_{S_n}^{(l,m)} - K_{S_{n'}}^{(l,m)}\|_\infty \to 0 \quad (n, n' \to \infty).$$

Hence, we find a function $K \in C^\infty(T^2)$ with $K_{S_n} \to K$ as $n \to \infty$ with respect to Fréchet topology on $C^\infty(T^2)$. Then the operator S defined by

$$S\xi(t) = \int_T K(t, s)\xi(s)ds \quad (\xi \in H^2,\, t \in T)$$

belongs to $\mathbb{K}^\infty$ and $S_n - S \to 0$ as $n \to \infty$ with respect to the seminorms, which implies the conclusion. It is clear by the above argument that $\mathbb{K}^\infty$ is a *-ideal of $\mathcal{T}^\infty$ and Fréchet closed.

We define a homomorphism $q : \mathcal{T}^\infty \to C^\infty(T)$ by $q(T_f + S) = f$, which is continuous with respect to the seminorms cited before. The following lemma is already clear:

Lemma 3.2. *We obtain the following short exact sequence as F^*-algebras:*

$$0 \longrightarrow \mathbb{K}^\infty \xrightarrow{\ i\ } \mathcal{T}^\infty \xrightarrow{\ q\ } C^\infty(T) \longrightarrow 0,$$

where i is the canonical inclusion.

We next deduce the following lemma, which is a smooth version of C^*-algebra case:

Lemma 3.3. *We have the following isomorphism:*

$$\mathbb{K}^\infty \simeq \varinjlim (M_n(\mathbb{C}), \varphi_n),$$

where the homomorphisms $\varphi_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ *are given by*

$$\varphi_n(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad (A \in M_n(\mathbb{C}),\ n \geq 1).$$

Proof. Let P_n ($n \geq 1$) be the orthogonal projections on H^2 defined by

$$\begin{aligned} P_n\xi(t) &= \sum_{k=0}^{n-1} \langle \xi \mid z^k \rangle z^k(t) \\ &= \sum_{k=0}^{n-1} \left(\int_T \xi(s) s^{-k} ds \right) t^k \\ &= \int_T \sum_{k=0}^{n-1} t^k s^{-k} \xi(s) ds \quad (\xi \in H^2), \end{aligned}$$

which implies that

$$K_{P_n}(t,s) = \sum_{k=0}^{n-1} t^k s^{-k}.$$

Then $P_n \mathbb{K}^\infty P_n$ is isomorphic to $M_n(\mathbb{C})$. Indeed, the kernel function $K_{P_n S P_n}$ for $S \in \mathbb{K}^\infty$ is calculated as follows: since

$$\begin{aligned} K_{SP_n}(t,s) &= \int_T K_S(t,r) K_{P_n}(r,s) dr \\ &= \int_T \sum_{k=0}^{n} r^k s^{-k} K_S(t,r) dr, \end{aligned}$$

we have that

$$\begin{aligned} K_{P_n S P_n}(t,s) &= \int_T K_{P_n}(t,u) K_{SP_n}(u,s) du \\ &= \int_T \sum_{k=0}^{n-1} t^k u^{-k} \left(\int_T \sum_{k'=0}^{n-1} r^{k'} s^{-k'} K_S(u,r) dr \right) du \\ &= \int_T \int_T \sum_{k,k'=0}^{n-1} t^k u^{-k} r^{k'} s^{-k'} K_S(u,r) dr du \\ &= \sum_{k,k'=0}^{n-1} t^k s^{-k'} \int_T \int_T r^{k'} u^{-k} K_S(u,r) dr du \\ &= \sum_{k,k'=0}^{n-1} c_{k,k'} t^k s^{-k'}, \end{aligned}$$

where

$$c_{k,k'} = \int_T \int_T r^{k'} u^{-k} K_S(u,r) dr du$$

are the Fourier coefficients of $K_S \in C^\infty(T^2)$.

On the other hand, we define the matrix units E_{ij} in what follows: when $i = j$, we define

$$E_{ii} = T_{z^{i-1}}T^*_{z^{i-1}} - T_{z^i}T^*_{z^i}.$$

For $i \neq j$, we define

$$E_{ij} = \begin{cases} T_{z^{j-i}}E_{ii} & (i < j) \\ E_{jj}T_{z^{j-i}} & (i > j). \end{cases}$$

It is not hard to see that $\{E_{ij}\}$ forms a family of matrix units. By taking $m = -n$ in the computation of the kernel function of $T_{z^n}T_{z^m} - T_{z^{n+m}}$, we have

$$K_{I-T_{z^n}T^*_{z^n}}(t,s) = \sum_{k=0}^{n-1} t^k s^{-k}.$$

Hence we have

$$K_{E_{ii}}(t,s) = t^{i-1}s^{-(i-1)}.$$

More generally, we obtain that

$$K_{E_{ij}}(t,s) = t^{j-1}s^{-(i-1)}.$$

Then $P_n\mathbb{K}^\infty P_n$ is generated by the matrix units $\{E_{ij}\}_{i,j=1}^n$ so that it is isomorphic to $M_n(\mathbb{C})$ with the seminorms given by

$$\|(\lambda_{kl})\|_{p,q} = \sup_{t,s\in T}\left|\sum_{k,l=0}^{n-1}\lambda_{kl}l^pk^qt^ls^{-k}\right| \quad ((\lambda_{kl}) \in M_n(\mathbb{C})).$$

For any $S \in \mathbb{K}^\infty$, $\|S - P_nSP_n\|_{l,m} \to 0$ as $n \to \infty$ for any $l, m \geq 0$ since $\{c_{k,k'}\}$ belongs to the Schwartz space on $\mathbb{Z}^2$. Therefore,

$$\|S - P_nSP_n\|_{l,m} \to 0 \quad (n \to \infty)$$

for any $l, m \in \mathbb{Z}_{\geq 0}$. Hence, the conclusion follows. □

By the above lemma, we deduce the following corollaries:

Corollary 3.4. *$\mathbb{K}^\infty$ is a simple F^*-algebra, which is equal to the commutator F^*-ideal $[\mathcal{T}^\infty, \mathcal{T}^\infty]$ of $\mathcal{T}^\infty$.*

In what follows, we study briefly the F^*-crossed products $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of $\mathcal{T}^\infty$ by the gauge action α_θ of $\mathbb{Z}$. Let α_θ be the action of $\mathbb{Z}$ on $\mathcal{T}^\infty$ defined by

$$\alpha_\theta(T_f) = T_{f_\theta} \quad (f \in C^\infty(T), n \in \mathbb{Z}),$$

where $f_\theta(z) = f(e^{2\pi i\theta}z)$, which gives a F^*-dynamical system $(\mathcal{T}^\infty, \mathbb{Z}, \alpha_\theta)$. We also consider the unitary operator U_θ on H^2 defined by

$$U_\theta\xi(t) = \xi(e^{2\pi i\theta}t) \quad (\xi \in H^2, t \in T).$$

It is easily seen that $U_\theta^*\xi(t) = U_\theta^{-1}\xi(t) = \xi(e^{-2\pi i\theta}t)$. Then we form the F^*-crossed products $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of the F^*-dynamical system $(\mathcal{T}^\infty, \mathbb{Z}, \alpha_\theta)$, which could be viewd

as the deformation quantization $(D^2 \times S^1)_\theta$ of the solid torus $D^2 \times S^1$. In fact, let $\mathcal{T}^\infty[\mathbb{Z}]$ be the *-algebra of all finite sums

$$f = \sum_{n\in\mathbb{Z},\, |n|\leq N} A_n U_\theta^n \quad (A_n \in \mathcal{T}^\infty,\, N \in \mathbb{Z}_{\geq 0}),$$

where its multiplication is determined by $U_\theta A U_\theta^{-1} = \alpha_\theta(A)$ and its *-operation is given by $(AU_\theta)^* = \alpha_\theta^{-1}(A^*)U_\theta^{-1}$. For $f = \sum A_n U_\theta^n \in \mathcal{T}^\infty[\mathbb{Z}]$, we induce the seminorms defined by

$$\|f\|_{p,q,r,s} = \sup_{n\in\mathbb{Z}} (1+|n|^2)^p \|A_n\|_{q,r,s} \quad (p,q,r,s \in \mathbb{Z}_{\geq 0}).$$

We define the F^*-crossed product $(D^2 \times S^1)_\theta = \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ by the completion of $\mathcal{T}^\infty[\mathbb{Z}]$ with respect to the seminorms cited above. For $S \in \mathbb{K}^\infty$, we calculate that

$$\begin{aligned}\alpha_\theta(S)\xi(t) &= U_\theta S U_\theta^* \xi(t)\\ &= U_\theta \int_T K_S(t,s)\xi(e^{-2\pi i\theta}s)ds\\ &= \int_T K_S(e^{2\pi i\theta}t, e^{2\pi i\theta}s)\xi(s)ds\end{aligned}$$

to obtain that

$$K_{\alpha_\theta(S)}(t,s) = K_S(e^{2\pi i\theta}t, e^{2\pi i\theta}s).$$

Therefore, we have $\alpha_\theta(\mathbb{K}^\infty) = \mathbb{K}^\infty$ so that we construct a F^*-dynamical system $(\mathbb{K}^\infty, \mathbb{Z}, \alpha_\theta)$. Since

$$\begin{aligned}K_{\alpha_\theta(P_n)}(t,s) &= K_{P_n}(e^{2\pi i\theta}t, e^{2\pi i\theta}s)\\ &= \sum_{k=0}^{n-1} (e^{2\pi i\theta}t)^k (e^{2\pi i\theta}s)^{-k}\\ &= \sum_{k=0}^{n-1} t^k s^{-k} = K_{P_n}(t,s),\end{aligned}$$

we have $\alpha_\theta(P_n\mathbb{K}^\infty P_n) = P_n\mathbb{K}^\infty P_n$. Therefore, we also construct F^*-dynamical systems $(P_n\mathbb{K}^\infty P_n, \mathbb{Z}, \alpha_\theta^{(n)})$, where $\alpha_\theta^{(n)}$ are the restrictions of α_θ on $P_n\mathbb{K}^\infty P_n$. Let i_n be the isomorphism from $P_n\mathbb{K}^\infty P_n$ onto $M_n(\mathbb{C})$ defined before and $i = \varinjlim i_n$ the isomorphism from $\varinjlim P_n\mathbb{K}^\infty P_n$ onto $\mathbb{K}^\infty$ induced by the isomorphisms i_n. We write by $\overline{\alpha}_\theta^{(n)}$ the action $i_n \circ \alpha_\theta \circ i_n^{-1}$.

Proposition 3.5. *We have the following isomorphism:*

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z}, \widetilde{\varphi}_n),$$

where $\widetilde{\varphi}_n$ are the inclusions induced naturally by φ_n.

Proof. Since $i \circ \overline{\alpha}_\theta^{(n)} = \alpha_\theta \circ i$ for any $n \geq 1$, we have

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \varinjlim (P_n\mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \varphi_n),$$

where $\varphi_n : P_n\mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z} \to P_{n+1}\mathbb{K}^\infty P_{n+1} \rtimes_{\alpha_\theta^{(n+1)}} \mathbb{Z}$ are the canonical inclusions. Moreover, since $i_n \circ \alpha_\theta^{(n)} = \overline{\alpha}_\theta^{(n)} \circ i_n$, we find isomorphisms

$$\psi_n : P_n\mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z} \xrightarrow{\simeq} M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z}.$$

Then since $\psi_n \circ \varphi_n = \widetilde{\varphi}_n \circ \psi_n$, we conclude

$$\varinjlim (P_n\mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \varphi_n) \simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z}, \widetilde{\varphi}_n)$$

as desired. $\square$

Then we construct a *-homomorphism

$$\rho_n : M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z} \to M_n(\mathbb{C})\hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}),$$

where $\mathcal{S}(\mathbb{Z})$ is the set of all rapidly decreasing sequences $\{c_n\} \subset \mathbb{C}$ and $\hat{\otimes}_\gamma$ means the tensor product of F^*-algebras completed by the topology induced by the seminorms defined by

$$\left\| \sum_{j=1}^N x_j \otimes y_j \right\|_{k,l} = \inf \sum_{j=1}^N \|x_j\|_k \|y_j\|_l,$$

where the infinimum is taken over the all representations of $\sum_{j=1}^N x_j \otimes y_j$. Equivalently, $M_n(\mathbb{C})\hat{\otimes}_\gamma S(\mathbb{Z})$ is regarded as $\mathcal{S}(\mathbb{Z}, M_n(\mathbb{C}))$ with the ordinary convolution as its product. For $x \in M_n(\mathbb{C}) \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}$, we define

$$\rho_n(x) = xU_\theta^n.$$

It is easily seen that it is an isomorphism. Moreover, since

$$\|\rho_n(x)\|_{p,q,r,s} = \sup_{m\in\mathbb{Z}} (1+m^2)^p \|x_m U_\theta^n\|_{q,r,s} = \|x\|_{p,q,r,s} \quad (p,q,r,s \in \mathbb{Z}_{\geq 0}),$$

for any $x = \sum x_m U_\theta^m \in \mathcal{T}^\infty\mathbb{Z}$, it is Fréchet isometry. Therefore, we have

$$M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z} \simeq M_n(\mathbb{C})\hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z})$$

by ρ_n. Now it is immediately known that the following fact follows:

Corollary 3.6. *The isomorphism*

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \mathbb{K}^\infty \hat{\otimes}_\gamma C^\infty(T)$$

holds.

Proof. By Proposition 3.5 and Lemma 3.3, we have that

$$\begin{aligned}
\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} &\simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\overline{\alpha}_\theta^{(n)}} \mathbb{Z}, \widetilde{\varphi}_n) \\
&\simeq \varinjlim (M_n(\mathbb{C})\hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}), \widetilde{\varphi}_n \otimes id_{\mathcal{S}(\mathbb{Z})}) \\
&\simeq \left(\varinjlim (M_n(\mathbb{C}), \varphi_n) \right) \hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}) \\
&\simeq \mathbb{K}^\infty \hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}).
\end{aligned}$$

Since $\mathcal{S}(\mathbb{Z})$ is isomorphic to $C^\infty(T)$ sending by the Fourier transform, the conclusion follows. $\square$

We end this section by stating the following fact:

Corollary 3.7. *We have the following short exact sequence:*

$$0 \longrightarrow \mathbb{K}^\infty \otimes_\gamma \mathcal{S}(\mathbb{Z}) \xrightarrow{\ \widetilde{i}\ } (D^2 \times S^1)_\theta \xrightarrow{\ \widetilde{q}\ } C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z} \longrightarrow 0,$$

where $\overline{\alpha}_\theta : C^\infty(T) \times \mathbb{Z} \to C^\infty(T)$ is the Fréchet continuous action defined by

$$\overline{\alpha}_\theta^n(f)(z) = f(e^{2\pi i n\theta} z) \quad (f \in C^\infty(T), z \in T),$$

with a bounded linear section $\widetilde{s}$ of $\widetilde{q}$.

Proof. Since $i \circ \alpha_\theta^n = \alpha_\theta^n \circ i$ and $q \circ \alpha_\theta^n = \overline{\alpha}_\theta^n \circ q$ for all $n \in \mathbb{Z}$, it is clear that the desired short exact sequence holds and $\widetilde{s}(fU_\theta^n) = T_f U_\theta^n$ ($f \in C^\infty(T)$, $n \in \mathbb{Z}$). $\square$

4. Metric Approximation Property

We introduce an analogue of the notion of metric approximation property for Banach spaces [4]. Let $\mathfrak{A}, \mathfrak{B}$ be two Banach spaces and $\mathfrak{I} \subset \mathfrak{B}$ an M-ideal. In [4], the authors prove that if $\mathfrak{A}$ is separable and has the metric approximation property, then each contractive map $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}$ has a lift $\widetilde{\varphi} : \mathfrak{A} \to \mathfrak{B}$ which is contractive and satisfies $q \circ \widetilde{\varphi} = \varphi$, where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$ is the quotient map. Our purpose in this section is to define this property for F^*-algebras to prove lifting problem cited above. The topology on $\mathfrak{A}$ induced by its seminorms $\{\|\cdot\|_k\}_{k\geq 0}$ is same as that induced by the metric $d_\mathfrak{A}$ defined by

$$d_\mathfrak{A}(a, b) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|a - b\|_k}{1 + \|a - b\|_k} \quad (a, b \in \mathfrak{A}).$$

We say that a linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is bounded if and only if there exists a constant $C > 0$ with

$$d_\mathfrak{B}(\varphi(a), 0) \leq C d_\mathfrak{A}(a, 0) \quad (a \in \mathfrak{A}).$$

Definition 4.1. *Let $\mathfrak{A}$ be a F^*-algebra and $\{\|\cdot\|_k\}_{k\geq 0}$ its seminorms. We say that it has the metric approximation property if there exists a family of bounded linear maps $\{\theta_n\}_{n\geq 1}$ on $\mathfrak{A}$ with the following properties:*

(1) *each θ_n has a finite rank,*
(2) *for any $a \in \mathfrak{A}$, $d_\mathfrak{A}(\theta_n(a), a) \to 0$ as $n \to \infty$.*

We give some examples of F^*-algebras with the metric approximation property. Here we note that $d_\mathfrak{A}(\theta_n(a), a) \to 0$ is satisfied if and only if $\|\theta_n(a) - a\|_k \to 0$ for any $k \geq 0$.

Example 4.2. For an integer $n \geq 2$, let $\mathbb{F}_n$ be the free group with n-generators. Given $g \in \mathbb{F}_n$, we denote by $|g|$ its word length, and for $f \in \mathbb{C}[\mathbb{F}_n]$ and an integer $k \in \mathbb{Z}_{\geq 0}$, we define seminorms by

$$\|f\|_k = \sup_{g \in \mathbb{F}_n} (1+|g|)^k |f(g)|.$$

The Schwartz space $\mathcal{S}(\mathbb{F}_n)$ is defined by the completion of $\mathbb{C}[\mathbb{F}_n]$ with respect to the above seminorms. For $f \in \mathcal{S}(\mathbb{F}_n)$, we define an bounded operator $\lambda(f)$ on the Hilbert space $l^2(\mathbb{F}_n)$ by the convolution with f, that is,

$$(\lambda(f)\xi)(g) = (f * \xi)(g) = \sum_{h \in \mathbb{F}_n} f(h)\xi(h^{-1}g) \quad (g \in \mathbb{F}_n, \xi \in l^2(\mathbb{F}_n)),$$

on which the seminorms are defined by

$$\|\lambda(f)\|_k = \|f\|_k \quad (k \in \mathbb{Z}_{\geq 0}).$$

This definition is well-defined. Indeed, if $\lambda(f) = 0$, then $\lambda(f)\delta_e = 0$, where $e \in \mathbb{F}_n$ is the unit and the element $\delta_e \in l^2(\mathbb{F}_n)$ is defined by

$$\delta_e(g) = \begin{cases} 1 & (g = e) \\ 0 & (g \neq e). \end{cases}$$

Hence, for any $g \in \mathbb{F}_n$, we have that

$$\begin{aligned} 0 &= (\lambda(f)\delta_e)(g) = (f * \delta_e)(g) \\ &= \sum_{h \in \mathbb{F}_n} f(h)\delta_e(h^{-1}g) = f(g). \end{aligned}$$

Therefore, $\lambda(f) = 0$ leads $f = 0$, which implies that the seminorms are well-defined. We define the F^*-algebra $C_r^*(\mathbb{F}_n)^\infty$ by the completion of the *-algebra generated by the bounded operators $\lambda(f)\,(f \in \mathcal{S}(\mathbb{F}_n))$. Here we claim that

$$C_r^*(\mathbb{F}_n)^\infty = \{\lambda(f) \mid f \in \mathcal{S}(\mathbb{F}_n)\}.$$

In fact, it is clear that $\lambda(f)^* = \lambda(f^*)$ for any $f \in \mathcal{S}(\mathbb{F}_n)$, where

$$f^*(g) = \overline{f(g^{-1})} \in \mathcal{S}(\mathbb{F}_n).$$

For any $T \in C_r^*(\mathbb{F}_n)^\infty$, there exits a family $\{\lambda(f_n)\}_{n \geq 1}\,(f_n \in \mathcal{S}(\mathbb{F}_n))$ which converges to T with respcet to the seminorms cited above. Then, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$\|f_n - f_m\|_k = \|\lambda(f_n) - \lambda(f_m)\|_k \to 0 \quad (n, m \to \infty),$$

which implies that there exists a function $f \in \mathcal{S}(\mathbb{F}_n)$ which is the limit of $\{f_n\}$. Thus, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$\|T - \lambda(f)\|_k \leq \|T - \lambda(f_n)\|_k + \|\lambda(f_n) - \lambda(f)\|_k \to 0 \quad (n \to \infty).$$

We construct a family of finite dimensional bounded linear maps $\{\theta_k\}$ on $C_r^*(\mathbb{F}_n)^\infty$ in what follows. Given an integer $k \geq 1$, let $E_k = \{g \in \mathbb{F}_n \,|\, |g| \leq k\}$ and χ_k the function on $\mathcal{S}(\mathbb{F}_n)$ defined by

$$\chi_k(g) = \begin{cases} 1 & (g \in E_k) \\ 0 & (g \notin E_k). \end{cases}$$

Since the number of elements of E_k is finite for each $k \geq 1$, the linear maps $\psi_n : \mathbb{F}_n \to \mathbb{F}_n$ defined by

$$\psi_k(\lambda(f)) = \lambda(\chi_k f)$$

have finite ranks. We define the finite linear bounded maps $\theta_k : \mathbb{F}_n \to \mathbb{F}_n$ $(k \geq 1)$ by

$$\theta_k(\lambda(f)) = \lambda(e^{-|\cdot|/k}\chi_k f).$$

Then for any $l \geq 0$ and $f \in C_r^*(\mathbb{F}_n)^\infty$, we compute that

$$\begin{aligned}
&\|\lambda(f) - \theta_k(\lambda(f))\|_l \\
&\leq \|\lambda(f) - \lambda(e^{-|\cdot|/k}f)\|_l + \|\lambda(e^{-|\cdot|/k}f) - \lambda(e^{-|\cdot|/k}\chi_k f)\|_l \\
&= \|f - e^{-|\cdot|/k}f\|_l + \|e^{-|\cdot|/k}f - e^{-|\cdot|/k}\chi_k f\|_l \\
&= \sup_{g \in \mathbb{F}_n} \left|(1+|g|)^l f(g)(1 - e^{-|g|/k})\right| + \sup_{g \in \mathbb{F}_n} \left|(1+|g|)^l f(g) e^{-|g|/k}(1 - \chi_k(g))\right| \\
&\leq \|f\|_l \sup_{g \in \mathbb{F}_n} \left|(1 - e^{-|g|/k})\right| + \sup_{|g| \geq k+1} \left|(1+|g|)^l f(g) e^{-|g|/k}\right| \\
&\to 0 \quad (k \to \infty).
\end{aligned}$$

Therefore, $C_r^*(\mathbb{F}_n)^\infty$ has the metric approximation property.

Example 4.3. According to [16], the smooth noncommutative 2-torus T_θ^2 is isomorphic to the Fréchet inductive limit

$$\varinjlim C^\infty(T)\hat{\otimes}_\gamma(M_{p_n}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C})).$$

We show that it also has the metric approximation property. As a preparation, we verify that the Fréchet algebra $C^\infty(T)\hat{\otimes}_\gamma M_q(\mathbb{C})$ has the metric approximation property. It suffices to show that $C^\infty(T)$ has this property since if it had this property with a family $\{\theta_n^{(q)}\}$ of bounded linear maps there, the family $\{\theta_n^{(q)} \otimes I_q\}$ would be the desired one for $C^\infty(T) \otimes M_q(\mathbb{C})$, where I_q is the identity map on $M_q(\mathbb{C})$. For $f \in C^\infty(T)$, we define the maps $\theta_n^{(q)} : C^\infty(T) \to C^\infty(T)$ by

$$\theta_n^{(q)}(f) = \sum_{|l| \leq n} \hat{f}(l) z^l \quad (n \geq 1),$$

where $\hat{f}(l)$ are the Fourier coefficients and $z \in C^\infty(T)$ is the canonical generator defined by $z(t) = t\,(t \in T)$. Then it is clear that they are of finite rank. For

$f \in C^\infty(T)$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\|f - \theta_n^{(q)}(f)\|_k &= \|f^{(k)} - (\theta_n(f))^{(k)}\|_\infty \\ &= \sup_{m \in \mathbb{Z}} \left| \widehat{f^{(k)}}(m) - \sum_{|l| \leq n} \hat{f}(l)(2\pi i l)^k \delta_l(m) \right| \\ &= \sup_{m \in \mathbb{Z}} \left| \sum_{|l| \geq n+1} \hat{f}(l)(2\pi i l)^k \delta_l(m) \right| \\ &= \sup_{m \in \mathbb{Z}, |m| \geq n+1} \left| \hat{f}(m)(2\pi m)^k \right| \\ &\to 0 \quad (n \to \infty),\end{aligned}$$

where $\delta_l(m) = 0\,(m \neq l), = 1(m = l)$, since $\{\hat{f}(l)\}_{l \in \mathbb{Z}}$ is a rapidly decreasing sequence by the hypothesis $f \in C^\infty(T)$. Hence $C^\infty(T)$ has the metric approximation property.

We turn to show briefly that T_θ^2 also has this property. For any $x \in T_\theta^2$, we define the sequence $\{x_n\}$ by

$$x_n = e_1^{(n)} x e_1^{(n)} + e_2^{(n)} x e_2^{(n)} \quad (n \geq 1),$$

where $e_j^{(n)}$ $(j = 1, 2)$ are the projections such that

$$e_1^{(n)} x e_1^{(n)} \in C^\infty(T) \otimes M_{p_n}(\mathbb{C}), \quad e_2^{(n)} x e_2^{(n)} \in C^\infty(T) \otimes M_{q_n}(\mathbb{C})$$

for any $x \in T_\theta^2$ ([16]). We define the linear maps Φ_n on T_θ^2 by

$$\Phi_n(x) = \theta_n^{(p_n)}(e_1^{(n)} x e_1^{(n)}) + \theta_n^{(q_n)}(e_2^{(n)} x e_2^{(n)})$$

It is easily seen that $\Phi_n(x) \to x$ with respect to the seminorms on T_θ^2 (see [16]), hence to the metric d as well. Therefore, T_θ^2 has the metric approximation property.

By the similar argument for $C^\infty(T)$, the operation of taking suspension preserves the metric approximation property.

Corollary 4.4. *If a F^*-algebra $\mathfrak{A}$ has the metric approximation property, so does its suspension $S^\infty\mathfrak{A}$.*

Proof. It suffices to show that the F^*-algebra

$$C_0^\infty(0,1) = \{f \in C^\infty(0,1) \,|\, f_+^{(n)}(0) = f_-^{(n)}(1) = 0\,(n \in \mathbb{Z}_{\geq 0})\}$$

has the metric approximation property. For any integer $j \geq 1$, we put

$$f_j(t) = e^{-\frac{1}{jt(1-t)}} \in C_0^\infty(0,1).$$

Let $\{\xi_j\}_{j=1}^\infty$ be the orthogonal family of $C_0^\infty(0,1)$ obtained by Schmidt orthogonalization of $\{f_j\}$. Then we define the linear maps $\theta_n : C_0^\infty(0,1) \to C^\infty(0,1)$

by

$$\theta_n(f)(t) = \sum_{j=1}^{n} \langle f|\xi_j\rangle \xi_j(t) \quad (\xi \in C_0^\infty(0,1),\, t \in (0,1),\, n \geq 1).$$

It is easily seen that the images of θ_n are included in $C_0^\infty(0,1)$. By the similar argument for $C^\infty(T)$, we obtain the conclusion. □

For a F^*-algebra $\mathfrak{A}$, by $\mathfrak{A}^*$ we denote the set of all bounded linear functionals on $\mathfrak{A}$, where we say a linear functional φ on $\mathfrak{A}$ is bounded if and only if

$$\|\varphi\| = \sup_{a \in \mathfrak{A}\setminus\{0\}} \frac{|\varphi(a)|}{d_{\mathfrak{A}}(a,0)} < \infty.$$

Before we proceed to show the lifting problem, we need the following lemma:

Lemma 4.5. *Let $\mathfrak{A}, \mathfrak{B}$ be two F^*-algebras. Suppose that $\mathfrak{I}$ is an F^*-ideal of $\mathfrak{B}$ and that L, N are finite dimensional subspaces of $\mathfrak{A}$ with $L \subset N$. We consider the following diagram of bounded linear maps*:

$$\begin{array}{ccccc}
L & \xrightarrow[\subset]{\iota} & N & \xrightarrow{\Psi} & \mathfrak{B} \\
 & & \Big\| & & \Big\downarrow q \\
L & \xrightarrow[\subset]{\iota} & N & \xrightarrow[\varphi]{} & \mathfrak{B}/\mathfrak{I},
\end{array}$$

where q and ι are the quotient map and the natural inclusion respectively, and suppose that

$$d_{\mathfrak{B}}(q \circ \Psi(a) - \varphi(a), 0) \leq \varepsilon d_{\mathfrak{A}}(a,0) \quad (a \in L).$$

for a positive constant $\varepsilon > 0$. Then there is a bounded linear map $\varphi' : N \to \mathfrak{B}/\mathfrak{I}$ with the property that

$$\begin{cases} \varphi = q \circ \varphi' \\ d_{\mathfrak{B}}(\varphi'(a), 0) \leq d_{\mathfrak{A}}(a,0) & (a \in N) \\ d_{\mathfrak{B}}(\varphi'(a) - \Psi(a), 0) \leq 6\varepsilon & (a \in L). \end{cases}$$

Proof. This lemma is an analogy of Lemma 2.5 in [4]. Let D' and K be the closed unit ball of $L\hat{\otimes}_\gamma\mathfrak{B}$ and $N\hat{\otimes}_\gamma\mathfrak{B}$ respectively, that is,

$$D' = \{\varphi : \mathfrak{B} \to L \,|\, \|\varphi\|_{L\hat{\otimes}_\gamma\mathfrak{B}^*} \leq 1\} \subset L\hat{\otimes}_\gamma\mathfrak{B}^*$$
$$K = \{\varphi : \mathfrak{B} \to N \,|\, \|\varphi\|_{N\hat{\otimes}_\gamma\mathfrak{B}^*} \leq 1\} \subset N\hat{\otimes}_\gamma\mathfrak{B}^*,$$

where

$$\|\varphi\|_{L\hat{\otimes}_\gamma\mathfrak{B}^*} = \sup_{a \in \mathfrak{B}\setminus\{0\}} \frac{d_{\mathfrak{A}}(\varphi(a),0)}{d_{\mathfrak{B}}(a,0)}$$

and $\|\cdot\|_{N\hat{\otimes}_\gamma\mathfrak{B}^*}$ is defined by the similar way for $\|\cdot\|_{L\hat{\otimes}_\gamma\mathfrak{B}^*}$, and $\mathrm{Aff}_T(D')$ the set of all affine functions ψ on D' such that $\psi(\alpha\varphi) = \alpha\psi(\varphi)$ for all $\alpha \in T, \varphi \in D'$. It is clear that $\mathfrak{I}$ is a M-ideal of $\mathfrak{B}$ and the equality

$$\mathfrak{B}^* = \mathfrak{I}^\perp \oplus \mathfrak{I}^*$$

holds as a linear space, where $\mathfrak{I}^\perp$ is the annihilator of $\mathfrak{I}$. Let $e : \mathfrak{B}^* \to \mathfrak{I}^\perp$ be the natural projection and W the image of $N\hat{\otimes}_\gamma\mathfrak{B}^*$ via $1 \otimes e$, which is equal to $N\hat{\otimes}_\gamma\mathfrak{I}^\perp$. Then D' is mapped weak* homeomorphically to $D \subset K$ through the natural embedding $\iota \otimes 1 : L\hat{\otimes}_\gamma\mathfrak{B}^* \to N\hat{\otimes}_\gamma\mathfrak{B}^*$. We may identify the closed unit ball of $N\hat{\otimes}_\gamma\mathfrak{B}^*$ with $F = K \cap W$. We also identify the closed unit ball of $L\hat{\otimes}_\gamma\mathfrak{B}^*$ with $D' \cup W'$, where $W' = (1 \otimes e)(L\hat{\otimes}_\gamma\mathfrak{B}^*) = L\hat{\otimes}_\gamma\mathfrak{I}^\perp$. It is verified by the same argument in the proof of Lemma 2.5 in [4] that

$$(\iota \otimes 1)(1 \otimes e) = (1 \otimes e)(\iota \otimes 1) \tag{1}$$

and

$$(\iota \otimes 1)(D' \cap W') = D \cap (\iota \otimes 1)(W') = D \cap W = D \cap F. \tag{2}$$

Thus we have the following diagram of restrictions

$$\begin{array}{ccc} \mathrm{Aff}_T(D \cap F) & \longleftarrow & \mathrm{Aff}_T(F) \\ \uparrow & & \uparrow \\ \mathrm{Aff}_T(D) & \longleftarrow & \mathrm{Aff}_T(K). \end{array} \tag{3}$$

Since $1 \otimes e : L\hat{\otimes}_\gamma\mathfrak{B}^* \to L\hat{\otimes}_\gamma\mathfrak{I}^\perp$ maps D' onto $D' \cap W$, D is mapped onto $D \cap F$ by (1) and (2) and D satisfies the condition of Lemma 2.1 in [4]. Therefore, with the diagram (3), we obtain the conclusion by the same argument of Lemma 2.5 in [4]. $\square$

Proposition 4.6. *Let $\mathfrak{A}, \mathfrak{B}$ be two F^*-algebras and $\mathfrak{I} \subset \mathfrak{B}$ an F^*-ideal. If $\mathfrak{A}$ is separable and has the metric approximation property, then for any bounded linear map $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}$, there exists a bounded linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ with the property that $q \circ \Phi = \varphi$, where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$ is the quotient map.*

Proof. This proof is inspired by that of Theorem 2.6 in [4]. We fix a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathfrak{A}$ dense in $\mathfrak{A}$. We construct recursively the pairs $\{(L_n, \theta_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ which consist of increasing finite dimensional subspaces $L_n \subset \mathfrak{A}$ with $a_n \in L_n$ for any $n \in \mathbb{Z}_{\geq 0}$ and bounded linear maps $\theta_n : \mathfrak{A} \to L_n$ with the property that for any $a \in L_{n-1}$, the inequalities

$$d_{\mathfrak{A}}(a, \theta_n(a)) \leq \frac{1}{2^n}$$

are satisfied. We put $L_0 = \{0\}$ and $\theta_0 = 0$. We suppose that for some $n \in \mathbb{Z}_{\geq 0}$ the pairs $(L_0, \theta_0), \cdots, (L_n, \theta_n)$ with the above properties are given. By the approximation property of $\mathfrak{A}$, there exists a bounded linear map $\theta_{n+1} : \mathfrak{A} \to \mathfrak{A}$ such that for each $a \in L_n$, the inequality

$$d_{\mathfrak{A}}(a, \theta_{n+1}(a)) \leq \frac{1}{2^{n+1}}$$

holds. We define the subspace L_{n+1} of $\mathfrak{A}$ by

$$L_{n+1} = L_n + \theta_{n+1}(\mathfrak{A}) + \mathbb{C}a_{n+1}.$$

Then we have the desired pairs $\{(L_n, \theta_n)\}_{n\in\mathbb{Z}_{\geq 0}}$. We note that $\cup_{n\in\mathbb{Z}_{\geq 0}} L_n$ is dense in $\mathfrak{A}$ with respect to Fréchet topology.

Next we inductively define a family of bounded linear maps

$$\Psi_n : L_n \to \mathfrak{B} \quad (n \in \mathbb{Z}_{\geq 0})$$

such that

$$q \circ \Psi_n(a) = \varphi(a) \quad (a \in L_n). \tag{4}$$

Putting $\Psi_0 = 0$, suppose that for some $n \in \mathbb{Z}_{\geq 0}$, bounded linear maps $\Psi_0, \cdots, \Psi_n$ satisfying (4) are constructed. Then we have that for any $a \in L_{n-1}$,

$$\begin{aligned} d_{\mathfrak{B}/\mathfrak{J}}(q \circ \Psi_n \circ \theta_n(a), \varphi(a)) &= d_{\mathfrak{B}/\mathfrak{J}}(\varphi \circ \theta_n(a), \varphi(a)) \\ &\leq C d_{\mathfrak{A}}(\theta_n(a), a) \\ &\leq \frac{C}{2^n}. \end{aligned}$$

By Lemma 4.5, we find a bounded map $\Psi_{n+1} : L_{n+1} \to \mathfrak{B}$ such that $\varphi = q \circ \Psi_{n+1}$ on L_{n+1} and that for any $a \in L_{n-1}$ with

$$d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n \circ \theta_n(a)) \leq \frac{6C}{2^n}.$$

Therefore, we compute that

$$\begin{aligned} d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) &\leq \frac{6C}{2^n} + d_{\mathfrak{B}}(\Psi_n(a), \Psi_n \circ \theta_n(a)) \\ &\leq \frac{6C}{2^n} + d_{\mathfrak{A}}(a, \theta_n(a)) \\ &\leq \frac{6C+1}{2^n} \quad (a \in L_{n-1}). \end{aligned}$$

Hence for a fixed integer $n_0 \in \mathbb{Z}_{\geq 0}$, we have for all $n \geq n_0$,

$$d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) \leq \frac{6C+1}{2^n}.$$

Thus, for a fixed integer $n_0 \in \mathbb{Z}_{\geq 0}$, the family of bounded linear maps $\{\Psi_n\}$ converges to some $\Psi^{(n_0)} : L_{n_0-1} \to \mathfrak{B}$. Therefore, we have the bounded linear map

$$\Psi : \bigcup_{n\in\mathbb{Z}_{\geq 0}} L_n \to \mathfrak{B}$$

such that $\Psi|_{L_n} = \Psi^{(n)}\, (n \in \mathbb{Z}_{\geq 0})$, and we can extend it to that on the closure of $\cup_{n\in\mathbb{Z}_{\geq 0}} L_n$ which is equal to $\mathfrak{A}$. This completes the proof. □

5. Mayer-Vietoris Exact Sequence

This section is devoted to proving Mayer-Vietoris exact sequence for the entire cyclic cohomology. We firstly give a short proof of Bott periodicity for the entire cyclic cohomology by using the following Meyer's excision for the entire cyclic theory [14]:

Proposition 5.1. *Let*

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{q} Q \longrightarrow 0$$

be a short exact sequence of F^*-algebras with a bounded linear section s of q. Then the following 6-terms exact sequence:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(Q) & \xrightarrow{q^*} & HE^{\mathrm{ev}}(P) & \xrightarrow{i^*} & HE^{\mathrm{ev}}(K) \\ \uparrow & & & & \downarrow \\ HE^{\mathrm{od}}(K) & \xleftarrow[i^*]{} & HE^{\mathrm{od}}(P) & \xleftarrow[q^*]{} & HE^{\mathrm{od}}(Q) \end{array}$$

holds.

This yields the following fact, which has been already shown by Brodzki and Plymen [3] using bivariant entire homology and cohomology theory:

Lemma 5.2 (Bott periodicity for entire cyclic cohomology)**.** *For a F^*-algebra $\mathfrak{A}$,*

$$HE^{\mathrm{ev}}(S^\infty\mathfrak{A}) \simeq HE^{\mathrm{od}}(\mathfrak{A}), \quad HE^{\mathrm{od}}(S^\infty\mathfrak{A}) \simeq HE^{\mathrm{ev}}(\mathfrak{A}).$$

Proof. By the exact sequence cited above, we have the following exact diagram:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(\mathfrak{A}) & \longrightarrow & HE^{\mathrm{ev}}(C^\infty\mathfrak{A}) & \longrightarrow & HE^{\mathrm{ev}}(\mathfrak{I}) \\ \uparrow & & & & \downarrow \\ HE^{\mathrm{od}}(\mathfrak{I}) & \longleftarrow & HE^{\mathrm{od}}(C^\infty\mathfrak{A}) & \longleftarrow & HE^{\mathrm{od}}(\mathfrak{A}). \end{array}$$

By Lemma 2.3, we deduce the conclusion. □

In what follows, we show an entire cyclic cohomology version of Mayer-Vietoris exact sequence. Before stating it, we review briefly the fibered product of F^*-algebras, which is an noncommutative analogue of the connected sum of two manifolds. Let $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{B}$ be F^*-algebras and $f_j : \mathfrak{A}_j \to \mathfrak{B} \quad (j = 1, 2)$ epimorphisms.

Definition 5.3. $\{(a_1, a_2) \in \mathfrak{A}_1 \oplus \mathfrak{A}_2 \mid f_1(a_1) = f_2(a_2)\}$ *is called the fibered product of* $(\mathfrak{A}_1, \mathfrak{A}_2)$ *along* (f_1, f_2) *over* $\mathfrak{B}$*, which we denote by* $\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2$*. Let g_j be the projections of* $\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2$ *onto* $\mathfrak{A}_j \, (j = 1, 2)$*.*

Theorem 5.4 (Mayer-Vietoris Exact Sequence for entire cyclic cohomology)**.** *In the situation of Definition 5.3, suppose that $\mathfrak{B}$ has the metric approximation property and separable. Then we have that the following exact diagram:*

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) & \longrightarrow & HE^{\mathrm{od}}(\mathfrak{B}) & \xrightarrow{-f_1^*+f_2^*} & HE^{\mathrm{od}}(\mathfrak{A}_1) \oplus HE^{\mathrm{od}}(\mathfrak{A}_2) \\ {\scriptstyle g_1^*+g_2^*}\uparrow & & & & \downarrow{\scriptstyle g_1^*+g_2^*} \\ HE^{\mathrm{ev}}(\mathfrak{A}_1) \oplus HE^{\mathrm{ev}}(\mathfrak{A}_2) & \xleftarrow[-f_1^*+f_2^*]{} & HE^{\mathrm{ev}}(\mathfrak{B}) & \longleftarrow & HE^{\mathrm{od}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) \end{array}$$

holds.

Proof. We write

$$C = \{(h_1, h_2) \in C^\infty\mathfrak{A}_1 \oplus C^\infty\mathfrak{A}_2 \mid f_1 \circ (h_1)_+^{(n)}(0) = (-1)^n f_2 \circ (h_2)_+^{(n)}(0) \; (n \in \mathbb{Z}_{\geq 0})\}$$

and define a map $q : C \to \mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2$ by

$$q(h_1, h_2) = (h_1(0), h_2(0)).$$

It is easily verified that the following sequence:

$$0 \longrightarrow \mathfrak{I} \xrightarrow{\ i\ } C \xrightarrow{\ q\ } \mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2 \longrightarrow 0$$

is exact, where

$$\mathfrak{I} = \{(h_1, h_2) \in C \mid h_j(0) = 0 \; (j = 1, 2)\}$$

and i is the canonical inclusion. Then there exists a bounded linear section s of q defined by

$$s(a_1, a_2) = ((1-t)a_1, (1-t)a_2). \quad ((a_1, a_2) \in \mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2, \; t \in [0, 1])$$

Then by Proposition 5.1, we have the following exact diagram:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) & \longrightarrow & HE^{\mathrm{ev}}(C) & \longrightarrow & HE^{\mathrm{ev}}(\mathfrak{I}) \\ \big\uparrow & & & & \big\downarrow \\ HE^{\mathrm{od}}(\mathfrak{I}) & \longleftarrow & HE^{\mathrm{od}}(C) & \longleftarrow & HE^{\mathrm{od}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2). \end{array}$$

Moreover, repeating the argument cited above, $\mathfrak{I}$ is smoothly homotopic to $S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$. More precisely, we define the map

$$r : \mathfrak{I} \to S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$$

by

$$r(h_1, h_2)(t) = (h_1(e^{1-1/t}), h_2(e^{1-1/t})) \quad ((h_1, h_2) \in C)$$

and let $i : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 \to \mathfrak{I}$ be the natural inclusion. It follows by the same argument discussed above that since the functions $t \mapsto h_j(e^{1-1/t})$ are in $S^\infty\mathfrak{A}_j$ $(j = 1, 2)$ and using the maps

$$\begin{aligned} &G_1 : \mathfrak{I} \to C^\infty([0, 1], \mathfrak{I}) \\ &G_2 : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 \to C^\infty([0, 1], S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2) \end{aligned}$$

defined by

$$(G_j)_s(h_1, h_2)(t) = (h_1(se^{1-1/t} + (1-s)t), h_2(se^{1-1/t} + (1-s)t)) \quad (j = 1, 2),$$

$\mathfrak{I}$ is smoothly homotopic to $S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$. Hence we conclude that

$$HE^*(\mathfrak{I}) \simeq HE^*(S^\infty\mathfrak{A}_1) \oplus HE^*(S^\infty\mathfrak{A}_2).$$

Now we define the map $\Psi : C \to S^\infty\mathfrak{B}$ by

$$\Psi(h_1, h_2)(t) = \begin{cases} f_1 \circ h_1(1-2t) & (t \in [0, 1/2]) \\ f_2 \circ h_2(2t-1) & (t \in [1/2, 1]). \end{cases}$$

We have to verify that it is well-defined. Since $f_1 \circ h_1(0) = f_2 \circ h_2(0)$ by the definition of C, it is continuous at $t = 1/2$. For $n = 1$, we compute that

$$\begin{aligned}\lim_{t\to 1/2+0} \frac{\Psi(h_1,h_2)(t) - \Psi(h_1,h_2)(1/2)}{t-1/2} &= \lim_{t\to 1/2+0} \frac{f_2 \circ h_2(2t-1) - f_2 \circ h_2(0)}{t-1/2} \\ &= f_2\left(\lim_{t\to 1/2+0} \frac{h_2(2t-1) - h_2(0)}{t-1/2}\right) \\ &= 2f_2\left(\lim_{\varepsilon\to 0+} \frac{h_2(\varepsilon) - h_2(0)}{\varepsilon}\right) \\ &= 2f_2 \circ (h_2)_+^{(1)}(0)\end{aligned}$$

and similarly, we compute that

$$\begin{aligned}\lim_{t\to 1/2-0} \frac{\Psi(h_1,h_2)(t) - \Psi(h_1,h_2)(1/2)}{t-1/2} &= \lim_{t\to 1/2-0} \frac{f_1 \circ h_1(1-2t) - f_1 \circ h_1(0)}{t-1/2} \\ &= -2f_1\left(\lim_{\varepsilon\to 0+} \frac{h_1(\varepsilon) - h_1(0)}{\varepsilon}\right) \\ &= -2f_1 \circ (h_1)_+^{(1)}(0) = 2f_2 \circ (h_2)_+^{(1)}(0).\end{aligned}$$

Thus $\Psi(h_1, h_2)$ is differentiable once at $t = 1/2$. Suppose that it is differentiable n-times at $t = 1/2$. Here we note that

$$\Psi^{(n)}(h_1, h_2)(t) = \begin{cases} (-2)^n f_1 \circ h_1^{(n)}(1-2t) & t \in (0, 1/2) \\ 2^n f_2 \circ h_2^{(n)}(2t-1) & t \in (1/2, 1) \end{cases}$$

and that

$$\Psi^{(n)}(h_1, h_2)(1/2) = (-2)^n f_1 \circ (h_1)_+^{(n)}(0) = 2^n f_2 \circ (h_2)_+^{(n)}(0)$$

by our hypothesis of induction. Then we compute that

$$\begin{aligned}&\lim_{t\to 1/2+0} \frac{\Psi(h_1,h_2)^{(n)}(t) - \Psi(h_1,h_2)^{(n)}(1/2)}{t-1/2} \\ &= 2^n \lim_{t\to 1/2+0} \frac{f_2 \circ h_2^{(n)}(2t-1) - f_2 \circ (h_2)_+^{(n)}(0)}{t-1/2} \\ &= 2^n f_2\left(\lim_{t\to 1/2+0} \frac{h_2^{(n)}(2t-1) - (h_2)_+^{(n)}(0)}{t-1/2}\right) \\ &= 2^n \cdot 2f_2\left(\lim_{\varepsilon\to 0+} \frac{h_2^{(n)}(\varepsilon) - (h_2)_+^{(n)}(0)}{\varepsilon}\right) \\ &= 2^{n+1} f_2 \circ (h_2)_+^{(n+1)}(0).\end{aligned}$$

Alternatively, we compute that

$$\begin{aligned}
&\lim_{t\to 1/2-0}\frac{\Psi(h_1,h_2)^{(n)}(t)-\Psi(h_1,h_2)^{(n)}(1/2)}{t-1/2}\\
&=(-2)^n\lim_{t\to 1/2-0}\frac{f_1\circ h_1^{(n)}(1-2t)-f_1\circ (h_1)_+^{(n)}(0)}{t-1/2}\\
&=(-2)^n\cdot(-2)f_1\left(\lim_{\varepsilon\to 0+}\frac{h_1^{(n)}(\varepsilon)-(h_1)_+^{(n)}(0)}{\varepsilon}\right)\\
&=(-2)^{n+1}f_1\circ(h_1)_+^{(n+1)}(0)=2^{n+1}f_2\circ(h_2)_+^{(n+1)}(0).
\end{aligned}$$

Therefore, $\Psi(h_1,h_2)$ is differentiable $(n+1)$-times for each $(h_1,h_2)\in C$, which ends the process of induction so that Ψ is well-defined. Since f_1 and f_2 are surjective, it is easily verified that Ψ is surjective. In fact, we canonically can lift them on $S^\infty\mathfrak{A}_j$, which are denoted by $f_j\,(j=1,2)$. Now given a $h\in S^\infty\mathfrak{B}$, we find $\widetilde{h_j}\in S^\infty\mathfrak{A}_j$ with

$$f_j(\widetilde{h_j}(t))=h(t)\quad (j=1,2).$$

Putting $h_1(t)=\widetilde{h_1}((1-t)/2)$, $h_2(t)=\widetilde{h_2}((1+t)/2)\,(0\le t\le 1)$, we then check that

$$\begin{aligned}
\Psi(h_1,h_2)(t)&=\begin{cases} f_1(h_1(1-2t)) & (0\le t\le 1/2)\\ f_2(h_2(2t-1)) & (1/2\le t\le 1)\end{cases}\\
&=\begin{cases} f_1\left(\widetilde{h_1}\left(1-2\dfrac{1-t}{2}\right)\right) & (0\le t\le 1/2)\\ f_2\left(\widetilde{h_2}\left(2\dfrac{1+t}{2}-1\right)\right) & (1/2\le t\le 1)\end{cases}\\
&=\begin{cases} f_1(\widetilde{h_1}(t)) & (0\le t\le 1/2)\\ f_2(\widetilde{h_2}(t)) & (1/2\le t\le 1)\end{cases}\\
&=h(t),
\end{aligned}$$

which implies that Ψ is surjective. As it is clear that its kernel is $C^\infty\mathfrak{I}_1\oplus C^\infty\mathfrak{I}_2$, where

$$\mathfrak{I}_j=\mathrm{Ker}f_j\quad (j=1,2),$$

we obtain the following short exact sequence:

$$0\longrightarrow C^\infty\mathfrak{I}_1\oplus C^\infty\mathfrak{I}_2\longrightarrow C\xrightarrow{\ \Psi\ }S^\infty\mathfrak{B}\longrightarrow 0. \tag{5}$$

Since $\mathfrak{B}$ has the metric approximation property, so does $S^\infty\mathfrak{B}$ by Corollary 4.4. Writing $\mathfrak{J}=C^\infty\mathfrak{I}_1\oplus C^\infty\mathfrak{I}_2$, the inverse map

$$\overline{\Psi}^{-1}:S^\infty\mathfrak{B}\to C/\mathfrak{J}$$

of the isomorphism $\overline{\Psi}$ induced by Ψ has a bounded lift

$$\widetilde{\overline{\Psi}}^{-1}:S^\infty\mathfrak{B}\to C$$

satisfying $\widetilde{\overline{\Psi}}^{-1} \circ q = \overline{\Psi}^{-1}$ by Proposition 4.6 since $\overline{\Psi}$ preserves each seminorms, where q is the quotient map from C onto $C/\mathfrak{J}$. Hence it is verified that $\widetilde{\overline{\Psi}}^{-1}$ is a bounded linear section of Ψ since we compute that

$$\Psi \circ \widetilde{\overline{\Psi}}^{-1} = \overline{\Psi} \circ q \circ \widetilde{\overline{\Psi}}^{-1} = \overline{\Psi} \circ \overline{\Psi}^{-1} = id_{S^\infty \mathfrak{B}}.$$

Therefore, we apply the above exact sequence (5) to Proposition 5.1 to obtain the following exact diagram:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(S^\infty \mathfrak{B}) & \longrightarrow & HE^{\mathrm{ev}}(C) & \longrightarrow & HE^{\mathrm{ev}}(C^\infty \mathfrak{J}_1 \oplus C^\infty \mathfrak{J}_2) \\ \uparrow & & & & \downarrow \\ HE^{\mathrm{od}}(C^\infty \mathfrak{J}_1 \oplus C^\infty \mathfrak{J}_2) & \longleftarrow & HE^{\mathrm{od}}(C) & \longleftarrow & HE^{\mathrm{od}}(S^\infty \mathfrak{B}). \end{array}$$

Since $HE^*(C^\infty \mathfrak{J}_1 \oplus C^\infty \mathfrak{J}_2) = 0$, we have that

$$HE^{\mathrm{ev}}(C) \simeq HE^{\mathrm{ev}}(S^\infty \mathfrak{B}) \simeq HE^{\mathrm{od}}(\mathfrak{B})$$
$$HE^{\mathrm{od}}(C) \simeq HE^{\mathrm{od}}(S^\infty \mathfrak{B}) \simeq HE^{\mathrm{ev}}(\mathfrak{B})$$

by the Bott periodicity (Lemma 5.2).

Summing up, we get the desired exact diagram in what follows:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) & \longrightarrow & HE^{\mathrm{od}}(\mathfrak{B}) & \longrightarrow & HE^{\mathrm{od}}(\mathfrak{A}_1) \oplus HE^{\mathrm{od}}(\mathfrak{A}_2) \\ \uparrow & & & & \downarrow \\ HE^{\mathrm{ev}}(\mathfrak{A}_1) \oplus HE^{\mathrm{ev}}(\mathfrak{A}_2) & \longleftarrow & HE^{\mathrm{ev}}(\mathfrak{B}) & \longleftarrow & HE^{\mathrm{od}}(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2). \end{array} \tag{6}$$

We consider the restriction $\Phi : S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2 \to S^\infty \mathfrak{B}$ of Ψ. We see that it is C^∞-homotopic to $\Pi : S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2 \to \mathfrak{B}$ defined by

$$\Pi(h_1, h_2)(t) = -\chi_{[0,1/2]}(t)(f_1 \circ h_1)(t) + \chi_{[1/2,1]}(t)(f_2 \circ h_2)(t)$$

for $(h_1, h_2) \in S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2, t \in [0,1]$. To see this, we note that for a Fréchet continuous homomorphism $f : \mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2 \to S^\infty \mathfrak{B}$, we have

$$\overline{f}^* = -f^* : HE^*(S^\infty \mathfrak{B}) \to HE^*(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2)$$

by [14], where $\overline{f} : \mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2 \to S^\infty \mathfrak{B}$ is the homomorphism defined by

$$\overline{f}(a)(t) = f(a)(1-t) \quad (a \in \mathfrak{A}, t \in [0,1]).$$

Indeed, we prepare the map $\Theta : S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2 \to C^\infty([0,1], S^\infty \mathfrak{B})$ defined by

$$\Theta_s(h_1, h_2)(t) = \begin{cases} f_1 \circ h_1(1 - 2t/(1+s)) & (0 \le t \le 1/2) \\ f_2 \circ h_2(2t/(1+s) - (1-s)/(1+s)) & (1/2 \le t \le 1). \end{cases}$$

so that it is a smooth homotopy between Ψ and the homomorphism given by

$$(h_1, h_2) \mapsto \left(t \mapsto \chi_{[0,1/2]}(t)(f_1 \circ h_1)(1-t) + \chi_{[1/2,1]}(t)(f_2 \circ h_2)(t)\right).$$

Therefore, we have the homotopy equivalence of Ψ and Π. Considering the following commutative diagram:

$$\begin{array}{ccc} HE^*(C) & \longrightarrow & HE^*(S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2) \\ \simeq \uparrow \Psi^* & & \| \\ HE^*(S^\infty\mathfrak{B}) & \xrightarrow[\Gamma^*=\Pi^*]{} & HE^*(S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2) \end{array}$$

we conclude that the right upper horizonal map and the left lower horizonal map in the diagram (6) are both $\Pi^* = -f_1^* + f_2^*$. Finally, since the following diagram

$$\begin{array}{ccc} HE^*(S^\infty\mathfrak{A}_1) \oplus HE^*(S^\infty\mathfrak{A}_2) & \longrightarrow & HE^*(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) \\ \simeq \downarrow & & \| \\ HE^*(\mathfrak{A}_1) \oplus HE^*(\mathfrak{A}_2) & \xrightarrow[g_1^*+g_2^*]{} & HE^*(\mathfrak{A}_1 \underset{\mathfrak{B}}{\#} \mathfrak{A}_2) \end{array}$$

is commutative, the vertical maps in the diagram (6) are both $g_1^* + g_2^*$. This completes the proof. $\square$

6. The Entire Cyclic Cohomology of Noncommutative 3-spheres

In [1], Heegaard-type quantum 3-spheres with 3-parameters are constructed as C^*-algebras. With their construction in mind, we define noncommutative 3-spheres in the framework of F^*-algebras as follows; given an irrational number θ with $0 < \theta < 1$, let T_θ^2 be the smooth noncommutative 2-torus with unitary generators u_θ, v_θ subject to $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. There exists an isomorphism $\gamma_\theta : T_{-\theta}^2 \to T_\theta^2$ satisfying

$$\gamma_\theta(u_{-\theta}) = v_\theta, \quad \gamma_\theta(v_{-\theta}) = u_\theta$$

by their universality. We consider the following two F^*-crossed products:

$$(D^2 \times S^1)_\theta = \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}, \quad (D^2 \times S^1)_{-\theta} = \mathcal{T}^\infty \rtimes_{\alpha_{-\theta}} \mathbb{Z}$$

defined before. We define two epimorphisms f_j $(j = 1, 2)$ such as

$$f_1 : (D^2 \times S^1)_\theta \to T_\theta^2, \quad f_2 : (D^2 \times S^1)_{-\theta} \to T_\theta^2$$

by $f_1 = \widetilde{q}_+$, $f_2 = \gamma_\theta \circ \widetilde{q}_-$, where $\widetilde{q}_\pm$ are the epimorphisms from $\mathcal{T}^\infty \rtimes_{\alpha_{\pm\theta}} \mathbb{Z}$ onto $C^\infty(T) \rtimes_{\overline{\alpha}_{\pm\theta}} \mathbb{Z} = T_{\pm\theta}^2$ respectively.

Definition 6.1. *Given an irrational number θ, the noncommutative 3-sphere S_θ^3 is defined by the fibered product $(D^2 \times S^1)_\theta \underset{T_\theta^2}{\#} (D^2 \times S^1)_{-\theta}$ of $((D^2 \times S^1)_\theta\, (D^2 \times S^1)_{-\theta})$ along (f_1, f_2) over T_θ^2.*

First of all, we compute the entire cyclic cohomology of $(D^2 \times S^1)_\theta$. We note that the isomorphism $C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z} \simeq T_\theta^2$ holds and that by Lemma 4.3 in [16], we have

$$HE^*(C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}) \simeq HE^*(T_\theta^2) = HP^*(T_\theta^2),$$

where HP^* is the functor of periodic cyclic cohomology. According to Connes [5], we know the generators of $HP^*(T^2_\theta)$ as follows:

$$HP^{\mathrm{ev}}(T^2_\theta) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau'_\theta],$$
$$HP^{\mathrm{od}}(T^2_\theta) = \mathbb{C}[\tau^{(1)}_\theta] \oplus \mathbb{C}[\tau^{(2)}_\theta],$$

where τ_θ is the unique normalized trace on T^2_θ and

$$\tau'_\theta(a_0, a_1, a_2) = \tau_\theta(a_0(\delta^{(1)}_\theta(a_1)\delta^{(2)}_\theta(a_2) - \delta^{(2)}_\theta(a_1)\delta^{(1)}_\theta(a_2)))$$
$$\tau^{(j)}_\theta(a_0, a_1) = \tau_\theta(a_0\delta^{(j)}_\theta(a_1)) \quad (j = 1, 2),$$

where $\delta^{(j)}_\theta$ are the derivations on T^2_θ such that

$$\delta^{(1)}_\theta(u_\theta) = 2\pi i u_\theta,\ \delta^{(1)}_\theta(v_\theta) = 0,\ \delta^{(2)}_\theta(u_\theta) = 0,\ \delta^{(2)}_\theta(v_\theta) = 2\pi i v_\theta.$$

Proposition 6.2.

$$HE^{\mathrm{ev}}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau'_\theta \circ \widetilde{q}], \quad HE^{\mathrm{od}}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau^{(1)}_\theta \circ \widetilde{q}].$$

Proof. We remember the following short exact sequence:

$$0 \longrightarrow \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \xrightarrow{\widetilde{i}} (D^2 \times S^1)_\theta \xrightarrow{\widetilde{q}} C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z} \longrightarrow 0$$

appeared in Corollary 3.7. Hence we apply the above exact sequence to Proposition 5.1 to obtain the following exact diagram:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}) & \xrightarrow{\widetilde{q}^*} & HE^{\mathrm{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{\widetilde{i}^*} & HE^{\mathrm{ev}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ HE^{\mathrm{od}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) & \xleftarrow[\widetilde{i}^*]{} & HE^{\mathrm{od}}((D^2 \times S^1)_\theta) & \xleftarrow[\widetilde{q}^*]{} & HE^{\mathrm{od}}(C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}). \end{array}$$

Alternatively, we have by Corollary 3.6 and [12] that

$$HE^*(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq HE^*(\mathbb{K}^\infty \hat{\otimes}_\gamma C^\infty(T))$$
$$= H^{\mathrm{DR}}_*(T; \mathbb{C}),$$

which implies that

$$HE^{\mathrm{ev}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}, \quad HE^{\mathrm{od}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}.$$

Therefore, we have the following exact diagram:

$$\begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\widetilde{q}^*} & HE^{\mathrm{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{\widetilde{i}^*} & \mathbb{C} \\ \uparrow & & & & \downarrow \\ \mathbb{C} & \xleftarrow[\widetilde{i}^*]{} & HE^{\mathrm{od}}((D^2 \times S^1)_\theta) & \xleftarrow[\widetilde{q}^*]{} & \mathbb{C}^2. \end{array} \tag{7}$$

We note that there exists an element $[(\widetilde{\psi}_{2k+1})] \in HE^{\mathrm{od}}((D^2 \times S^1)_\theta)$ with the property that

$$(\widetilde{\psi}_{2k+1}) = (\widetilde{\psi}, 0, 0, \cdots),$$

and

$$B\widetilde{\psi} = \tau_\theta \circ \widetilde{q}, \quad b\widetilde{\psi} = 0,$$

where $b, B = AB_0$ are the operations defined by Connes [5]. Indeed, we define $\widetilde{\psi}$ by

$$\widetilde{\psi}(x, y) = \tau_\theta \circ \widetilde{q}(x\widetilde{\delta_\theta}^{(2)}(y)) \quad (x, y \in (D^2 \times S^1)_\theta),$$

where $\widetilde{\delta_\theta}^{(2)}$ is the derivation on $(D^2 \times S^1)_\theta$ induced by

$$\widetilde{\delta_\theta}^{(2)}\left(\sum_{n\in\mathbb{Z}} A_n U_\theta^n\right) = \sum_{n\in\mathbb{Z}} 2\pi i\theta n A_n U_\theta^n$$

for any $\sum_{n\in\mathbb{Z}} A_n U_\theta^n \in \mathcal{T}^\infty[\mathbb{Z}]$. We note that $\widetilde{\delta_\theta}^{(2)}$ is Fréchet continuous since

$$\begin{aligned}
\left\|\widetilde{\delta_\theta}^{(2)}\left(\sum_{n\in\mathbb{Z}} A_n U_\theta^n\right)\right\|_{p,q,r,s} &= \sup_{n\in\mathbb{Z}}(1+n^2)^p \left\|2\pi i\theta n A_n\right\|_{q,r,s} \\
&\leq 2\pi\theta \sup_{n\in\mathbb{Z}}(1+n^2)^{p+1} \left\|A_n\right\|_{q,r,s} \\
&= 2\pi\theta \left\|\sum_{n\in\mathbb{Z}} A_n U_\theta^n\right\|_{p+1,q,r,s}.
\end{aligned}$$

for any $p, q, r, s \in \mathbb{Z}_{\geq 0}$. In this case, let $1 \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ be the unit. It is clear that $\widetilde{\delta_\theta}^{(2)}(1) = 0$. Then by the definition of b and B, we have that

$$\begin{aligned}
B\widetilde{\psi}(x) &= \widetilde{\psi}(1, x) + \widetilde{\psi}(x, 1) \\
&= \tau_\theta \circ \widetilde{q}(x\widetilde{\delta_\theta}^{(2)}(1)) + \tau_\theta \circ \widetilde{q}(1\widetilde{\delta_\theta}^{(2)}(x)) \\
&= \tau_\theta \circ \widetilde{q}(\widetilde{\delta_\theta}^{(2)}(x)) \quad (x \in (D^2 \times S^1)_\theta).
\end{aligned}$$

We note that for any $f \in C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$,

$$\tau_\theta(f) = \int_T f(0)(t)dt.$$

Thus, we obtain that

$$\begin{aligned}
\tau_\theta \circ \widetilde{q}(\widetilde{\delta_\theta}(x)) &= \int_T \widetilde{q}(\widetilde{\delta_\theta}(x))(0)(t)dt \\
&= \int_T q(x(0))(t)dt = \tau_\theta \circ \widetilde{q}(x)
\end{aligned}$$

for any $x \in (D^2 \times S^1)_\theta$, which implies that $\widetilde{q}^*[\tau_\theta] = 0$. Hence, $\ker \widetilde{q}^* \neq 0$ so that the left vertical map of (7) is not 0, therefore, injective.

Similarly, we show that the right vertical map is also injective. Since θ is an irrational number, the set $\{e^{2\pi i\theta n} \in \mathbb{C} \,|\, n \in \mathbb{Z}\}$ is dense in T. Hence, for all $r \in [0, 1]$, there exists an sequence $\{N_j\}_j \subset \mathbb{Z}$ with $|\{\theta N_j\} - r| \to 0$ as $j \to \infty$, where

$$\{x\} = x - \max_{x\geq k,\, k\in\mathbb{Z}} k \quad (x \in \mathbb{R}).$$

We consider the family $\{U_{\theta N_j}\}$ of unitary operators on H^2. Since we see that for any $\xi \in H^2$,

$$\begin{aligned}\|(U_{\theta N_j} - U_{\theta N_k})\xi\|_{H^2}^2 &= \|(U_\theta^{N_j} - U_\theta^{N_k})\xi\|_{H^2}^2 \\ &= \int_T |\xi(e^{2\pi i\theta N_j}t) - \xi(e^{2\pi i\theta N_k}t)|^2 dt \\ &= \int_T |\xi(e^{2\pi i\theta(N_j - N_k)}t) - \xi(t)|^2 dt \to 0 \quad (j,k \to \infty)\end{aligned}$$

by the Lebesgue dominated convergence theorem, we obtain that $\{U_{\theta N_j}\}$ has the strong limit U_r. It is easily seen that $U_r\xi(t) = \xi(e^{2\pi ir}t)$ ($\xi \in H^2$, $t \in T$). Moreover, we define the operator h_θ on H^2 by

$$h_\theta\xi(t) = 2\pi\sum_{j=0}^\infty \{j\theta\}c_j t^j \quad \left(\xi(t) = \sum_{j=0}^\infty c_j t^j \in H^2\right).$$

Since $0 \le \{j\theta\} \le 1$, it is easily verified that h_θ is a bounded self-adjoint positive operator on H^2 and $U_{\theta r} = e^{irh_\theta}$ for $r \in [0,1]$ by Stone's theorem. Taking again a family $\{N_j\}_{j\in\mathbb{Z}_{\ge 0}} \subset \mathbb{Z}$ with $|e^{2\pi i\theta N_j} - e^{2\pi ir}| \to 0$ as $j \to \infty$, we have that

$$\begin{aligned}&\|\alpha_{\theta N_j}(x)\xi - \alpha_{\theta N_k}(x)\xi\|_{H^2} \\ &= \|U_{\theta N_j}xU_{-\theta N_j}\xi - U_{\theta N_k}xU_{-\theta N_k}\xi\|_{H^2} \\ &\le \|U_{\theta N_j}x(U_{-\theta N_j} - U_{-\theta N_k})\xi\|_{H^2} + \|(U_{\theta N_j} - U_{\theta N_k})xU_{-\theta N_k}\xi\|_{H^2} \\ &\to 0 \quad (x \in \mathcal{T}^\infty,\ \xi \in H^2)\end{aligned}$$

since the operation of product is strongly continuous. Therefore, it follows that $\alpha_r(x) = U_r x U_{-r}$ for $x \in \mathbb{B}(H^2)$. We write

$$\widetilde{\delta_\theta}^{(1)}(x) = h_\theta x - xh_\theta = \mathrm{ad}(h_\theta)(x) \quad (x \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z})$$

so that

$$e^{ir\widetilde{\delta_\theta}^{(1)}} = e^{ir\mathrm{ad}(h_\theta)} = \alpha_{\theta r} \quad (r \in [0,1]).$$

We now extend the homomorphism $\widetilde{q} : \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \to C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$ to that from the strong closure of $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ onto that of $C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$ faithfully acting on $L^2(T)$ because of the simplicity of $T_\theta^2 = C^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$, that is, that from $\mathbb{B}(H^2)$ onto $L^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$. We also extend the trace τ_θ on T_θ^2 to that on $L^\infty(T) \rtimes_{\overline{\alpha}_\theta} \mathbb{Z}$. We use the same letters for their extensions. Then, we have that $\widetilde{q} \circ \widetilde{\delta_\theta}^{(1)} = \delta_\theta^{(1)} \circ \widetilde{q}$ on $\mathbb{B}(H^2)$. Under the above preparation, we define the linear functional φ_0 on $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ by

$$\varphi_0(a) = -\tau_\theta \circ \widetilde{q}(ah_\theta) \quad (a \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}).$$

Then we compute that

$$\begin{aligned}(b\varphi_0)(a,b) &= \varphi_0(ab) - \varphi_0(ba) \\ &= -\tau_\theta \circ \widetilde{q}(abh_\theta) + \tau_\theta \circ \widetilde{q}(bah_\theta) \\ &= -\tau_\theta(\widetilde{q}(a)\widetilde{q}(b)\widetilde{q}(h_\theta)) + \tau_\theta(\widetilde{q}(b)\widetilde{q}(a)\widetilde{q}(h_\theta)) \\ &= \tau_\theta(\widetilde{q}(a)\widetilde{q}(h_\theta)\widetilde{q}(b) - \widetilde{q}(a)\widetilde{q}(b)\widetilde{q}(h_\theta)) \\ &= \tau_\theta(\widetilde{q}(a)\widetilde{q}(h_\theta b - bh_\theta)) \quad (a,b \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}).\end{aligned}$$

By the definition of $\widetilde{\delta_\theta}^{(1)}$ and $\widetilde{q} \circ \widetilde{\delta_\theta}^{(1)} = \delta_\theta^{(1)} \circ \widetilde{q}$, we have that

$$\begin{aligned}(b\varphi_0)(a,b) &= \tau_\theta(\widetilde{q}(a)\widetilde{q} \circ \widetilde{\delta_\theta}^{(1)}(b)) \\ &= \tau_\theta(\widetilde{q}(a)\delta_\theta^{(1)} \circ \widetilde{q}(b)) = (\tau_\theta^{(1)} \circ \widetilde{q})(a,b)\end{aligned}$$

for any $a,b \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$. Therefore, we obtain that

$$(b+B)[(\varphi_0, 0, \cdots)] = [(\tau_\theta^{(1)} \circ \widetilde{q}, 0, \cdots)],$$

which means that $[\tau_\theta^{(1)} \circ \widetilde{q}] = 0 \in HE^{\mathrm{od}}(\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z})$. Hence, we have $\ker \widetilde{q}^* \neq 0$ so that the right vertical map of (7) is also injective.

Summing up, we obtain the following exact diagram:

$$\begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\widetilde{q}^*} & HE^{\mathrm{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{0} & \mathbb{C} \\ \uparrow & & & & \downarrow \\ \mathbb{C} & \xleftarrow[0]{} & HE^{\mathrm{od}}((D^2 \times S^1)_\theta) & \xleftarrow[\widetilde{q}^*]{} & \mathbb{C}^2. \end{array} \tag{8}$$

to conclude that

$$HE^*((D^2 \times S^1)_\theta) \simeq \mathbb{C}^2/\mathbb{C} \simeq \mathbb{C}$$

as required. Moreover, we easily seen that $\widetilde{q}^* \neq 0$. Hence $\widetilde{q}^*[\tau'_\theta] = [\tau'_\theta \circ \widetilde{q}]$ and $\widetilde{q}^*[\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \widetilde{q}]$ are the generators of corresponding entire cyclic cohomology. □

We need the following lemma to end up the main result:

Lemma 6.3. *We have the following equalities:*

(1) $\tau_\theta \circ \gamma_\theta = \tau_{-\theta}$ *and* $\tau'_\theta \circ \gamma_\theta = -\tau'_{-\theta}$,
(2) $\tau_\theta^{(1)} \circ \gamma_\theta = \tau_{-\theta}^{(2)}$ *and* $\tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)}$.

Proof. Since $\tau_\theta \circ \gamma_\theta$ is a normalized trace on $T^2_{-\theta}$, it follows by uniqueness that $\tau_\theta \circ \gamma_\theta = \tau_{-\theta}$. We firstly verify that

$$\delta_\theta^{(1)} \circ \gamma_\theta = \delta_{-\theta}^{(2)}, \quad \delta_\theta^{(2)} \circ \gamma_\theta = \delta_{-\theta}^{(1)}.$$

In fact, it is sufficient to verify these equalities for generators. We compute that

$$\delta_\theta^{(j)} \circ \gamma_\theta(u_{-\theta}) = \delta_\theta^{(j)}(v_\theta) = \begin{cases} 0 & (j=1) \\ 2\pi i v_\theta & (j=2) \end{cases}$$

$$\delta_\theta^{(j)} \circ \gamma_\theta(v_{-\theta}) = \delta_\theta^{(j)}(u_\theta) = \begin{cases} 2\pi i u_\theta & (j=1) \\ 0 & (j=2). \end{cases}$$

We then deduce that

$$\begin{aligned} &\tau'_\theta(\gamma_\theta(b_0), \gamma_\theta(b_1), \gamma_\theta(b_2)) \\ &= \tau_\theta(\gamma_\theta(b_0)((\delta_\theta^{(1)} \circ \gamma_\theta(b_1))(\delta_\theta^{(2)} \circ \gamma_\theta(b_2)) - (\delta^{(2)} \circ \gamma_\theta(b_1))(\delta_\theta^{(1)}\gamma_\theta(b_2)))) \\ &= \tau_\theta(\gamma_\theta(b_0(\delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2) - \delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2))) \\ &= -\tau_\theta \circ \gamma_\theta(b_0(\delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2) - \delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2))) \\ &= -\tau_{-\theta}(b_0(\delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2) - \delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2))) \quad (b_0, b_1, b_2 \in T^2_{-\theta}). \end{aligned}$$

Moreover, for $b_0, b_1 \in T^2_{-\theta}$, we calculate that

$$\begin{aligned} \tau_\theta^{(1)} \circ \gamma_\theta(b_0, b_1) &= \tau_\theta^{(1)}(\gamma_\theta(b_0)\delta_\theta^{(1)}(\gamma_\theta(b_1))) \\ &= \tau_\theta^{(1)}(\gamma_\theta(b_0\delta_{-\theta}^{(2)}(b_1))) \\ &= \tau_{-\theta}^{(2)}(b_0, b_1). \end{aligned}$$

Similarly we have that $\tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)}$. $\square$

Under the above preparation, we determine the entire cyclic cohomology of noncommutative 3-spheres S^3_θ. By Theorem 5.4, we have the following exact diagram:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(S^3_\theta) & \longrightarrow & HE^{\mathrm{od}}(T^2_\theta) & \xrightarrow{-f_1^*+f_2^*} & G^1_\theta \oplus G^1_{-\theta} \\ {\scriptstyle g_1^*+g_2^*}\uparrow & & & & \downarrow{\scriptstyle g_1^*+g_2^*} \\ G^0_\theta \oplus G^0_{-\theta} & \xleftarrow[-f_1^*+f_2^*]{} & HE^{\mathrm{ev}}(T^2_\theta) & \longleftarrow & HE^{\mathrm{od}}(S^3_\theta), \end{array}$$

where $G^0_{\pm\theta} = HE^{\mathrm{ev}}((D^2 \times S^1)_{\pm\theta}), G^1_{\pm\theta} = HE^{\mathrm{od}}((D^2 \times S^1)_{\pm\theta})$ respectively. By Proposition 6.2 and the description in its proof, the above diagram becomes the following one:

$$\begin{array}{ccccc} HE^{\mathrm{ev}}(S^3_\theta) & \longrightarrow & \mathbb{C}^2 & \xrightarrow{-f_1^*+f_2^*} & \mathbb{C}^2 \\ {\scriptstyle g_1^*+g_2^*}\uparrow & & & & \downarrow{\scriptstyle g_1^*+g_2^*} \\ \mathbb{C}^2 & \xleftarrow[-f_1^*+f_2^*]{} & \mathbb{C}^2 & \longleftarrow & HE^{\mathrm{od}}(S^3_\theta). \end{array}$$

We describe precisely the maps $-f_1^* + f_2^*$ to compute $HE^*(S^3_\theta)$. For the even case, we check the map

$$-f_1^* + f_2^* : HP^{\mathrm{ev}}(T^2_\theta) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau'_\theta] \to \mathbb{C}[\tau'_\theta \circ \widetilde{q}] \oplus \mathbb{C}[\tau'_{-\theta} \circ \widetilde{q}] = G^0_\theta \oplus G^0_{-\theta}.$$

We have $f_1^*[\tau_\theta] = [\tau_\theta \circ \widetilde{q}] = 0$ by the calculation in Proposition 6.2 and $f_1^*[\tau'_\theta] = [\tau'_\theta \circ \widetilde{q}]$. Alternatively, it follows from Lemma 6.3 that $f_2^*[\tau_\theta] = [\tau_{-\theta} \circ \widetilde{q}] = 0$ by the same reason for the case of f_1^* and that $f_2^*[\tau'_\theta] = [\tau'_\theta \circ \widetilde{q}] = -[\tau'_{-\theta} \circ \widetilde{q}]$ by Lemma 6.3. On the other hand, for the odd case, we consider the map

$$-f_1^* + f_2^* : HP^{\mathrm{od}}(T_\theta^2) = \mathbb{C}[\tau_\theta^{(1)}] \oplus \mathbb{C}[\tau_\theta^{(2)}] \to \mathbb{C}[\tau_\theta^{(2)} \circ \widetilde{q}] \oplus \mathbb{C}[\tau_{-\theta}^{(2)} \circ \widetilde{q}] = G_\theta^1 \oplus G_{-\theta}^1.$$

Similarly we compute that

$$\begin{aligned} f_1^*[\tau_\theta^{(2)}] &= [\tau_\theta^{(2)} \circ \widetilde{q}] \\ f_1^*[\tau_\theta^{(1)}] &= [\tau_\theta^{(1)} \circ \widetilde{q}] = 0 \\ f_2^*[\tau_\theta^{(1)}] &= [\tau_\theta^{(1)} \circ \gamma_\theta \circ \widetilde{q}] = [\tau_{-\theta}^{(2)} \circ \widetilde{q}] \end{aligned}$$

and

$$f_2^*[\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \gamma_\theta \circ \widetilde{q}] = [\tau_{-\theta}^{(1)} \circ \widetilde{q}] = 0$$

by Lemma 6.3.

Therefore, we have the following exact diagram:

$$\begin{tikzcd} HE^{\mathrm{ev}}(S_\theta^3) \arrow[r, "0"] & \mathbb{C}^2 \arrow[r, "(\lambda,\mu)\mapsto(-\mu,\lambda)"] & \mathbb{C}^2 \arrow[d, "0"] \\ \mathbb{C}^2 \arrow[u] & \mathbb{C}^2 \arrow[l, "(\lambda,\mu)\mapsto(-\mu,-\mu)"'] & HE^{\mathrm{od}}(S_\theta^3), \arrow[l] \end{tikzcd}$$

by which we conclude that

$$\begin{aligned} HE^{\mathrm{ev}}(S_\theta^3) &\simeq \operatorname{coker}\{\mathbb{C} \oplus \mathbb{C} \ni (\lambda,\mu) \mapsto (-\mu,-\mu) \in \mathbb{C} \oplus \mathbb{C}\} \simeq \mathbb{C}, \\ HE^{\mathrm{od}}(S_\theta^3) &\simeq \ker\{\mathbb{C} \oplus \mathbb{C} \ni (\lambda,\mu) \mapsto (-\mu,-\mu) \in \mathbb{C} \oplus \mathbb{C}\} \simeq \mathbb{C}. \end{aligned}$$

This completes our computation of the entire cyclic cohomology of noncommutative 3-spheres.

Theorem 6.4. *The entire cyclic cohomology of noncommutative* 3*-spheres is isomorphic to the d'Rham homology of the ordinary* 3*-spheres with complex coefficients.*

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